**The gradient of secants and tangents to a graph**

Consider a function *f* : **R** *→* **R** and its graph *y = f* (*x*), which is a curve in the plane. We wish to find the *gradient* of this curve at a point. But first we need to define properly what we mean by the gradient of a curve at a point!

The module *Coordinate geometry* defines the gradient of a line in the plane: Given a

non-vertical line and two points on it, the **gradient** is defined as

run

rise

Gradient of a line.

Now, given a *curve* defined by *y = f* (*x*), and a point *p* on the curve, consider another point *q* on the curve near *p*, and draw the line *pq* connecting *p* and *q*. This line is called a **secant line**.

We write the coordinates of *p* as (*x*,*y*), and the coordinates of *q* as (*x +*∆*x*,*y +*∆*y*). Here ∆*x* represents a small change in *x*, and ∆*y* represents the corresponding small change in *y*.

*x*

*y*

0

x

*x +*

∆

*x*

*y*

=

*f*

(

*x*

)

*p*

= (

*x*

,

*y*

)

∆

*x*

*q*

= (

*x*

+ ∆

*x, y +*

∆

*y*

)

∆

*y*

=

*f*

(

*x*

+ ∆

*x) − f*

(

*x*

)

Secant connecting points on the graph *y = f* (*x*) at *x* and *x +*∆*x*.

As ∆*x* becomes smaller and smaller, the point *q* approaches *p*, and the secant line *pq* approaches a line called the **tangent** to the curve at *p*. We define the **gradient of the curve** at *p* to be the gradient of this tangent line.

*y*

*x*

0

x

*y*

=

*f*

(

*x*

)

*p*

= (

*x*

,

*y*

)

Secants on *y = f* (*x*) approaching the tangent line at *x*.

Note that, in this definition, the approximation of a tangent line by secant lines is just like the approximation of instantaneous velocity by average velocities, as discussed in the *Motivation* section.

With this definition, we now consider how to compute the gradient of the curve *y = f* (*x*) at the point *p =* (*x*,*y*).

Taking *q =* (*x +*∆*x*,*y +*∆*y*) as above, the secant line *pq* has gradient

Note that the symbol ∆ on its own has no meaning: ∆*x* and ∆*y* refer to change in *x* and *y*, respectively. You cannot cancel the ∆’s!

As ∆*x →* 0, the gradient of the tangent line is given by

We also denote this limit by

The notation *dx* indicates the instantaneous rate of change of *y* with respect to *x*, and is not a fraction. For our purposes, the expressions *dx* and *dy* have no meaning on their own, and the *d*’s do not cancel!

The gradient of a secant is analogous to average velocity, and the gradient of a tangent is analogous to instantaneous velocity. Velocity is the instantaneous rate of change of position with respect to time, and the gradient of a tangent to the graph *y = f* (*x*) is the instantaneous rate of change of *y* with respect to *x*